tain with a slice removed from its center. The mountain will have unit height before removing the slice, whenever the modal damping ratios, ζ_j and ζ_k are equal. The width of the slice, measured parallel to either axis, is equal to the difference between frequencies of two adjacent modes. The further apart adjacent modes are, the broader the slice and the lower the portion of the mountain remaining. The degree of modal coupling corresponds to the height of the remaining portion. From this illustration it is apparent that for widely separated modes there is negligible modal coupling *even when the off-diagonal terms of* ξ *are not small compared to the diagonal*

Intuitively, one may reason that as long as damping forces in the structure are small compared to inertia and stiffness forces (light damping) they will be important only to the extent that they dissipate energy over a period of time, and will not couple structural modes which are well-separated in frequency because of high cross-modal impedance. On the other hand, when two or more modal frequencies are closely spaced, cross-modal impedance is low. Inertia and stiffness forces tend to be balanced so that small damping forces are comparable to the small differences between inertia or stiffness forces.

Modal Separation Criteria

One may neglect coupling between two modes whenever the corresponding element of $\tilde{Z}_n(i\Omega)$ is small compared to unity. This condition may be stated, dropping the argument $i\Omega$ from $\tilde{Z}_n(i\Omega)$,

$$|e_k^T \tilde{Z}_n e_j| << 1 \tag{8}$$

where e_j denotes the jth column of an identity matrix. From (7a),

$$\begin{aligned} e_k^T \tilde{Z}_n e_j &= e_k Z_d^{-\frac{1}{2}} Z_n Z_d^{-\frac{1}{2}} e_j \\ &= (e_k^T Z_d^{-\frac{1}{2}} e_k)(e_k^T Z_n e_j)(e_j^T Z_d^{-\frac{1}{2}} e_j) \end{aligned} \tag{9}$$

It is recalled[4,5] that $\xi_{jj} = 2\zeta_j\omega_j$ so that

$$e_j^T Z_d^{-\frac{1}{2}} e_j = [(\omega_j^2 - \Omega^2) + i2\zeta_j\omega_j\Omega]^{-\frac{1}{2}} \tag{10}$$

and that

$$e_k^T Z_n e_j = i\Omega\xi_{kj} \tag{11}$$

where ξ_{kj} is the kjth element of the modal damping matrix for $j \neq k$. The largest value of $e_k \tilde{Z}_n e_j$ is realized whenever Ω equals either ω_j or ω_k. Without loss of generality, it may be assumed that $\Omega = \omega_j < \omega_k$. Then

$$e_j^T Z_d^{-\frac{1}{2}} e_j = (i2\zeta_j\omega_j^2)^{-\frac{1}{2}} = (i2\zeta_j\omega_j^2)^{-1}(i2\zeta_j\omega_j^2)^{\frac{1}{2}} \tag{12}$$

Furthermore,

$$i\omega_j\xi_{kj}(i2\zeta_j\omega_j^2)^{-1} = \xi_{kj}/2\zeta_j\omega_j = \xi_{kj}/\xi_{jj} \tag{13}$$

Substitution of (10) through (13) back into (9), with $\Omega = \omega_j$, leads to

$$e_k^T \tilde{Z}_n e_j = \left[\frac{i2\zeta_j\omega_j^2}{(\omega_k^2 - \omega_j^2) + i2\zeta_k\omega_j\omega_k}\right]^{\frac{1}{2}} \left(\frac{\xi_{kj}}{\xi_{jj}}\right)$$

Finally,

$$|e_k^T \tilde{Z}_n e_j| = \left\{\frac{2\zeta_j}{[(\beta^2 - 1)^2 + 4\zeta_k^2\beta^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \left|\frac{\xi_{kj}}{\xi_{jj}}\right| \tag{14}$$

where $\beta \equiv \omega_k/\omega_j > 1$. Neglecting the ζ_k^2 term in (14), one finds that (8) is satisfied if

$$[2\zeta_j/(\beta^2 - 1)]^{\frac{1}{2}} << 1 \tag{15}$$

provided that $\xi_{kj}/\xi_{jj} \sim 1$ or smaller. For example, if $\zeta_j = 0.015$ and $\beta = 2$, the left-hand-side of (15) equals 0.1.

Conclusions

It is concluded from this derivation that the degree of coupling between two classical normal modes depends not only on the ratio of the off-diagonal to diagonal terms of the modal damping matrix, but also on the percent of critical damping in the two modes and their frequency separation. The higher the percent of critical damping, the greater the frequency separation must be to uncouple the equations. Even when the classical normal modes do not diagonalize the damping matrix, the equations of motion will be uncoupled for all practical purposes, provided that adequate frequency separation exists between the modes.

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Stress Concentration in the Plastic Range

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Nomenclature

ϵ	=	strain or strain at discontinuity
ϵ_p	=	strain away from discontinuity
E	=	elastic modulus
$\bar{E}_s$, E_s, E_p	=	secant moduli
K_{el}	=	elastic stress concentration factor
K_p	=	plastic stress concentration factor
K_ϵ	=	plastic strain concentration factor
p	=	stress applied away from discontinuity
p_e	=	"equivalent elastic" stress applied away from discontinuity
σ	=	stress or stress at discontinuity
σ_y	=	yield stress in simple tension
σ_e	=	"equivalent elastic" stress at discontinuity

Introduction

STRUCTURAL components used in industry very often have discontinuities at which stress and strain

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concentrations occur. In the elastic range of material behavior, stress concentration factors have been established for many geometric configurations under different loading conditions. Relatively little is known about stress and strain concentration factors if the material at the discontinuity enters the plastic range.

The present Note proposes a simple semigraphical method of predicting stress and strain concentration factors when the stresses enter the plastic range. The procedure is based on a knowledge of only the stress-strain curve in uniaxial tension and the value of the elastic stress concentration factor. The method is applicable regardless of geometry and loading (assumed to be monotically increasing), since these effects are contained in the value of the elastic stress concentration factor.

Some results obtained with this semigraphical method are compared with specific analytical results, obtained by others, and show good agreement.

Semigraphical Method

The proposed method of finding the plastic stress and strain concentration factors at a discontinuity is based on the relation

$$K_{el}^2 = K_p K_\epsilon \tag{1}$$

This equation has been derived by Neuber[1] and it can be shown that a modification leads to[2]

$$(K_p / K_{el})^2 = E_s / E \tag{2}$$

In the foregoing the following definitions apply:

a) the elastic stress concentration factor is

$$K_{el} = \sigma_e / p_e \tag{3}$$

b) the plastic stress concentration factor is

$$K_p = \sigma / p \tag{4}$$

c) the plastic strain concentration factor is

$$K_\epsilon = \epsilon / \epsilon_p \tag{5}$$

It is assumed, that most materials have a stress-strain relation in uniaxial tension which can be described by

$$\epsilon = (\sigma / E) \; [1 + (3/7) \; (\sigma / \sigma_y)^{m-1}] \tag{6}$$

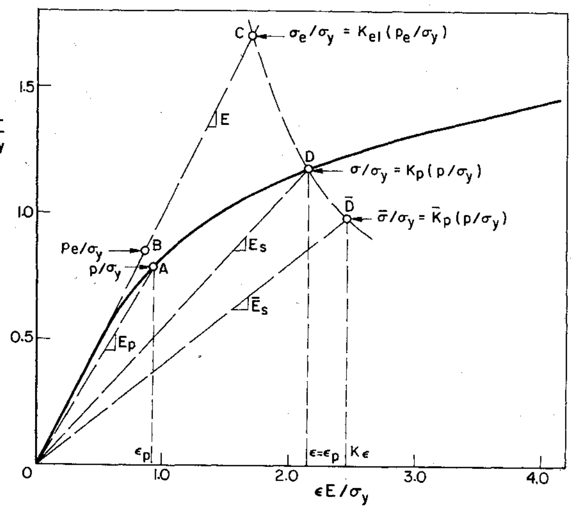

Fig. 1 Typical stress-strain diagram.

It is possible to derive an expression for K_p from Eqs. (1-6). The solution of the resulting equation however is obtained only by a lengthy trial and error procedure. For this reason a direct semigraphical method of solution is proposed.

Figure 1 shows a uniaxial stress-strain curve for a representative material. For an applied stress p (point A) away from the discontinuity, it is required to find the stress and strain concentration factors at the discontinuity, if the elastic stress concentration factor K_{el} is known. For the justification of using the uniaxial stress-strain curve for multiaxial stress states, the reader is referred to Ref. 3.

To apply the elastic stress concentration factor K_{el} it is first necessary to find the "equivalent elastic" applied stress p_e (point B). This is done by using Eq. (2) in the form

$$p_e = p(E_p / E)^{-1/2} \tag{7}$$

Application of Eq. (3) results at point C, in

$$\sigma_e = K_{el} p_e \tag{8}$$

Using Eq. (2) for an arbitrarily chosen secant modulus $\bar{E}_s$ one obtains

$$\bar{K}_p = K_{el} (\bar{E}_s / E)^{1/2} \tag{9}$$

and from this point $\bar{D}$,

$$\bar{\sigma} = \bar{K}_p p \tag{10}$$

Since $CD\bar{D}$ in Fig. 1 represents a hyperbola, some additional points of the type $\bar{D}$ may be required to draw this curve. These points $\bar{D}$ are found by repeating the calculations from Eq. (9) for several arbitrarily chosen moduli $\bar{E}_s$.

The next step consists of determining σ (point D) at the intersection of the stress-strain curve and curve $CD\bar{D}$ and the calculation of

$$K_p = \sigma / p \tag{11}$$

as the required plastic stress concentration factor at the discontinuity.

The corresponding plastic strain concentration factor then is obtained from Eq. (1) as

$$K_\epsilon = K_{el}^2 / K_p \tag{12}$$

The procedure is an extension, i.e., a two-step application of Neuber's Theorem[1] as illustrated in Ref. 4. In a typical application the designer usually has available the values of K_{el}, p_e, and E and therefore can, with the preceding method, determine K_p and K_ϵ. As K_{el} increases (e.g., very sharp

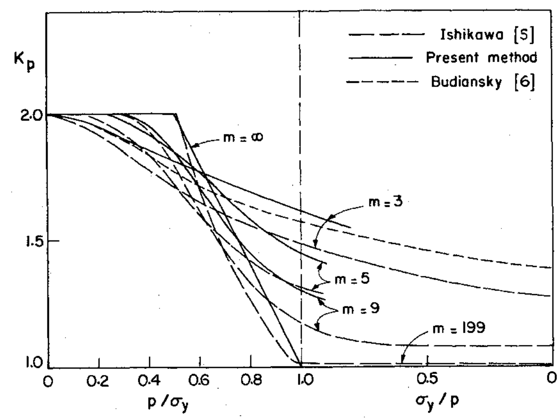

Fig. 2 Comparison of K_p obtained from proposed semigraphical method and rigorous analytical methods.

notches, etc.) this approximate method of determining K_p and K_ϵ may become increasingly more inaccurate and its usage should be restricted to relatively small and moderately small strain ranges.

Discussion

Recently Ishikawa[5] has examined the problem of plastic stress concentration around a hole in an infinite sheet under equal biaxial tension. His anlaysis makes use of the deformation theory of plasticity and the analytical results given include plastic stress concentration factors and stress fields.

The present method is capable of predicting only plastic stress and strain concentration factors at the discontinuity, but is not restricted to a particular value of K_{el}. The method therefore is only applicable for studying the local concentrations at the discontinuity.

For a value of $K_{el}=2$ we have shown in Fig. 2 a comparison between results of the proposed simigraphical method and results obtained by analytical methods[5,6] for K_p. The agreement between K_p values at various levels of applied stress is quite good. The maximum difference between our results and those of Ref. 5 is of the order of 10%. The agreement with the values of Ref. 6 is somewhat better.

Conclusions

1) It is suggested that the approach proposed in this note is quite general, since the type of loading and geometry are contained in the value of K_{el}. 2) Experimental verification would be necessary to establish the limitations of the proposed method.

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A Practical Aspect of Bass' Algorithm for Stabilizing Linear-Time-Invariant Systems

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IN the design of modern flight control systems whose dynamic behavior is described by the vector differential equation

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{1}$$

it is often required to compute a full-state feedback control law of the form

$$u(t)=Kx(t) \tag{2}$$

such that $(A+BK)$ is a stability matrix. A particular case of such a requirement is the solution of the steady-state Riccati equation by Newton's method[1] for aircraft with relaxed static stability.

Kleinman[2] and Bass[3] have proposed algorithms to generate stabilizing control laws. Bass' algorithm, however, has the advantage of simplicity and computational ease. The Bass algorithm calls for a solution of a Lyapunov type matrix equation. This is stated in the following theorem.[3]

Theorem—Bass Algorithm:

Let A, B be a controllable pair, then

$$K=-B'Z^{-1} \tag{3}$$

stabilizes the system (1) where $Z=Z'>0$ satisfies

$$-(A+\beta I_n)Z+Z[-(A+\beta I_n)]'=-2BB' \tag{4}$$

with $\beta>\|A\|$, where $\|\cdot\|$ is any vector-induced matrix norm,[4] and $(\cdot)'$ indicates the transpose.

The purpose of this Engineering Note is to provide a simple and computationally tractable value for the scalar β directly in terms of the elements of the A matrix. Obviously, if $\beta>\|A\|$, then the matrix $-(A+\beta I_n)$ is a stability matrix; therefore, any value of β which makes $-(A+\beta I_n)$ a stability matrix will satisfy the requirement for the Bass algorithm.

Let P be the modal matrix of A. Then A is given by

$$A=P\Lambda P^{-1} \tag{5}$$

where

$$\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\ldots\lambda_n) \tag{6}$$

With no loss of generality, let λ_1 be the eigenvalue of A with the largest absolute value real part, namely,

$$|Re(\lambda_1)|\geq|Re(\lambda_i)|\,\forall i \qquad i\neq 1 \tag{7}$$

Therefore, if $\beta>|Re(\lambda_1)|$, the matrix $-(A+\beta I_n)$ is a stability matrix. Applying the Gersgorin Theorem,[4] we readily obtain

$$|Re(\lambda_i)|\leq\max\left|a_{ii}+\left[\mathrm{sgn}(a_{ii})\right]\left(\sum_{j,j\neq i}|a_{ij}|\right)\right| \tag{8}$$

a_{ij} is the (i,j) element of the matrix A. Consequently, β in Eq. (4) can now be specified as

$$\beta=\eta\max\left|a_{ii}+\left[\mathrm{sgn}(a_{ii})\right]\left(\sum_{j,j\neq i}|a_{ij}|\right)\right| \tag{9}$$

where $\eta>1$.

A simple method has thus been established to determine the parameter β in the Bass algorithm. Equation (9) is amenable to simple machine calculations.

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